Chapter 3
Stability Analysis

3.1 Routh-Hurwitz Criterion

3.2 Root-Locus Technique

3.3 Nyquist Stability Criterion

3.4 Bode Plot and Stability

Stability of Control Systems

- A system is ***stable*** if finite input produces a bounded or finite output.
- For example for every step input applied to a system the out put must be finite.
- When it is subjected to an impulse input the output dies away to zero as time goes infinity.
- If the output tends to infinity as t approaches infinity, then the system is ***unstable***.
- If the output does not die out to zero or increase to infinity but tends to some finite but non-zero value then the system is said to be ***critically or marginally stable***.

Poles and Zeros

- The closed-loop transfer function G(s), generally be represented by:

$$G(s) = \frac{k(s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + ... + a_1 s + a_o}{(s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + ... + b_1 s + b_o}$$

 Equivalenty

$$G(s) = \frac{k(s+z_1)(s+z_2)....(s+z_m)}{(s+p_1)(s+p_2)....(s+p_n)}$$

Where: $z_1, z_2, ...,z_m$ are <u>zeros</u>

- values of s for which G(s) = 0

$p_1, p_2, ...,p_n$ are <u>poles</u>

- values of s for which G(s) = ∞

k - Gain of the system

- Generally poles and zeros can be real or complex.

$$s = \sigma + j\omega$$

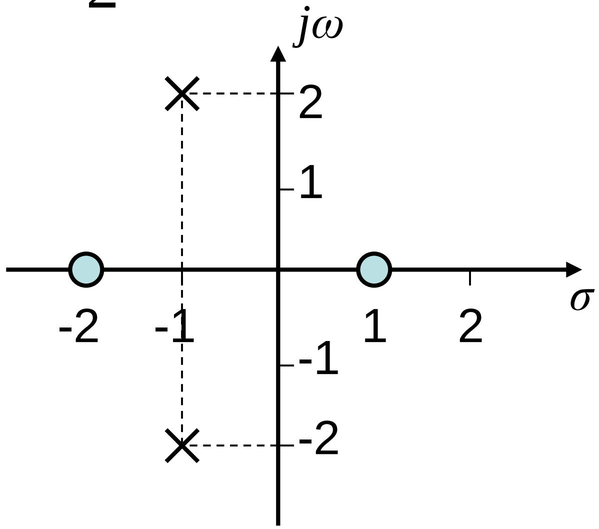
Examples:

$$a)\ \frac{s-1}{s^2-4s+4} = \frac{s-1}{(s-2)^2} \Rightarrow \text{poles are +2 and +2 ; zero +1}$$

$$b)\ \frac{1}{s^2+s+1};\ s_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm j\sqrt{4-1}}{2} = -0.5 \pm j0.87$$

Pole-zero plots

- Poles are marked with ‘x’ and zeros with ‘o’



Examples:

$$1)\ G(s)=\frac{1}{s+2}\Rightarrow\theta_o(s)=\frac{1}{s+2}\theta_i(s)$$

$$\text{if }\theta_i(s)=1\Rightarrow\theta_o(s)=\frac{1}{s+2}\Rightarrow\theta_o(t)=e^{-2t}$$

$$2)\ G(s)=\frac{1}{s-2}\Rightarrow\theta_o(s)=\frac{1}{s-2}\theta_i(s)$$

$$\text{if }\theta_i(s)=1\Rightarrow\theta_o(s)=\frac{1}{s-2}\Rightarrow\theta_o(t)=e^{2t}$$

$$3)\ G(s)=\frac{1}{[s-(-2+j)][s-(-2-j)]}\Rightarrow\theta_o(s)=G(s)\theta_i(s)$$

$$\text{if }\theta_i(s)=1\Rightarrow\theta_o(s)=G(s)\Rightarrow\theta_o(t)=e^{-2t}\sin t$$

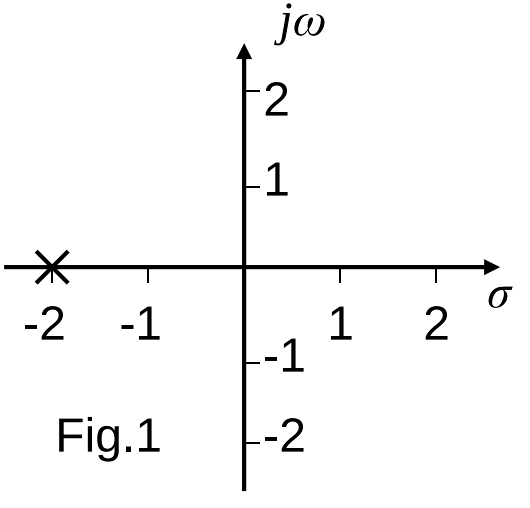


Fig.1

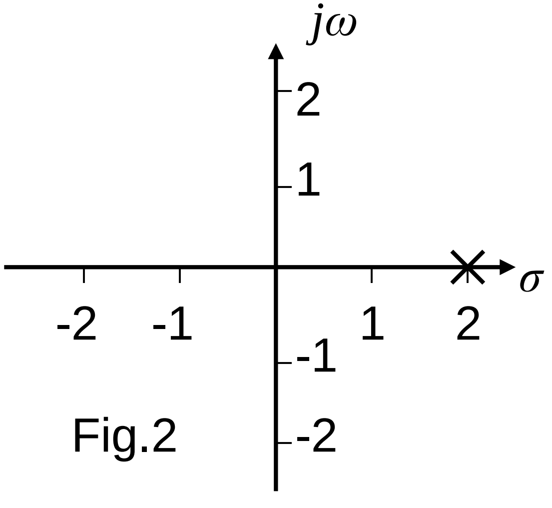


Fig.2

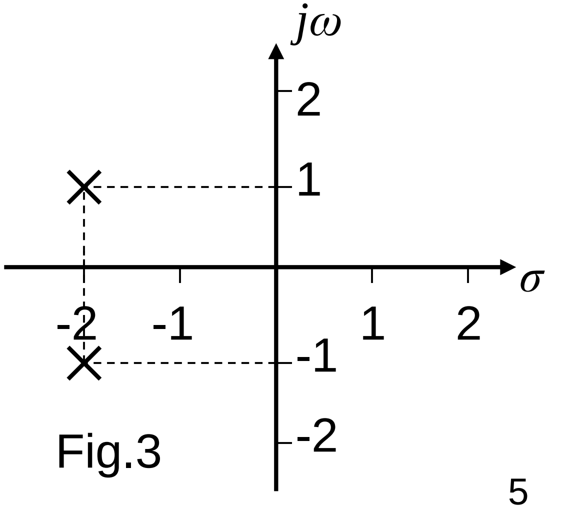
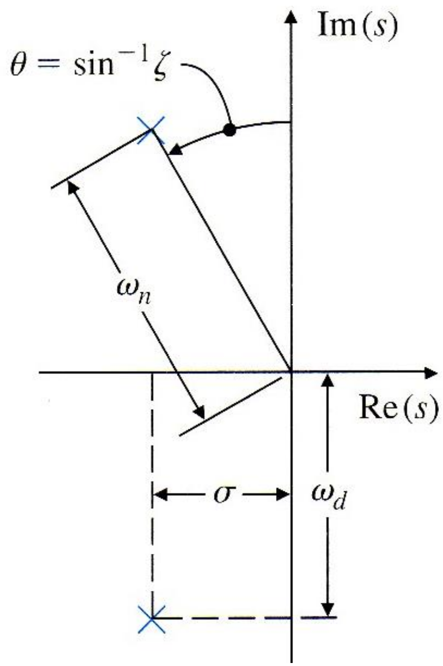


Fig.3

- In general when an impulse is applied to a system the output is in the form of the summation of a number of exponential terms.
- The following is possible to happen.
  - If just one of the exponential terms is of an exponential growing type, the system is ***unstable***
    - (i.e if any one of the poles has a +ve real part)
  - If all the poles are on the left side of the pole-zero plot or s-plane, then the system is ***stable.***
  - If just one pole is in the right hand side it is ***unstable.***
  - If one or more poles lie on the vertical axis of a pole-zero plot, then it is ***Critically stable.***

Summary 1

s-plane plot for a pair of complex poles



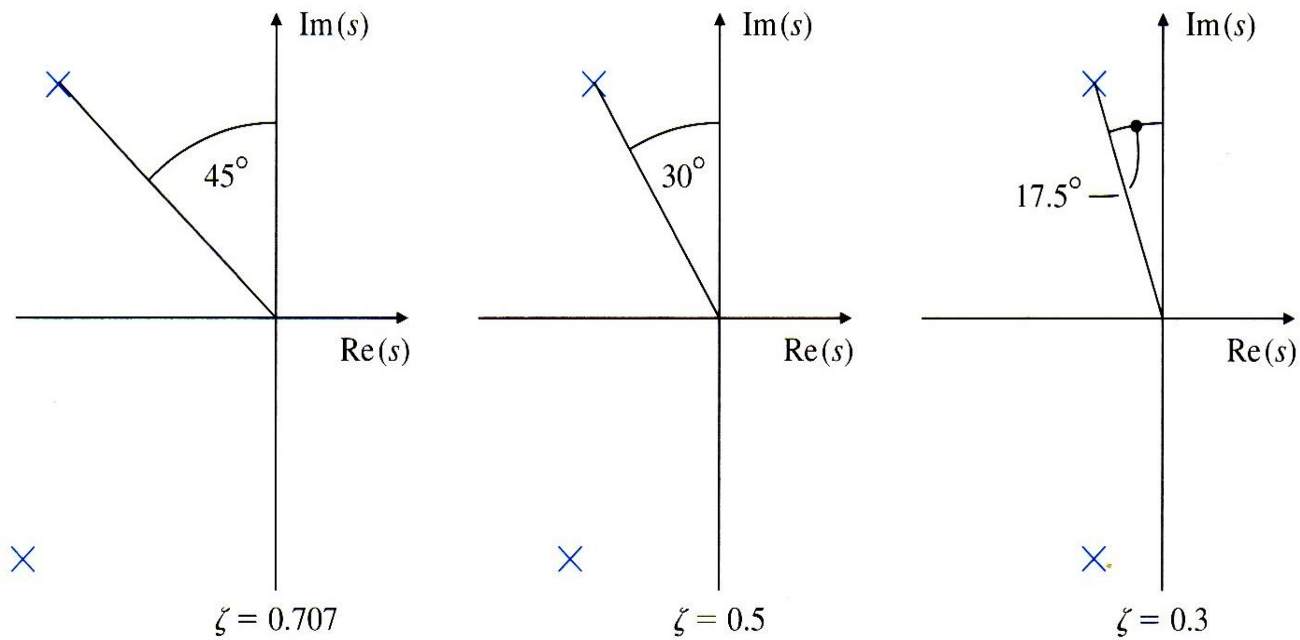


Figure: Pole location corresponding to three values of ς

Summary 2

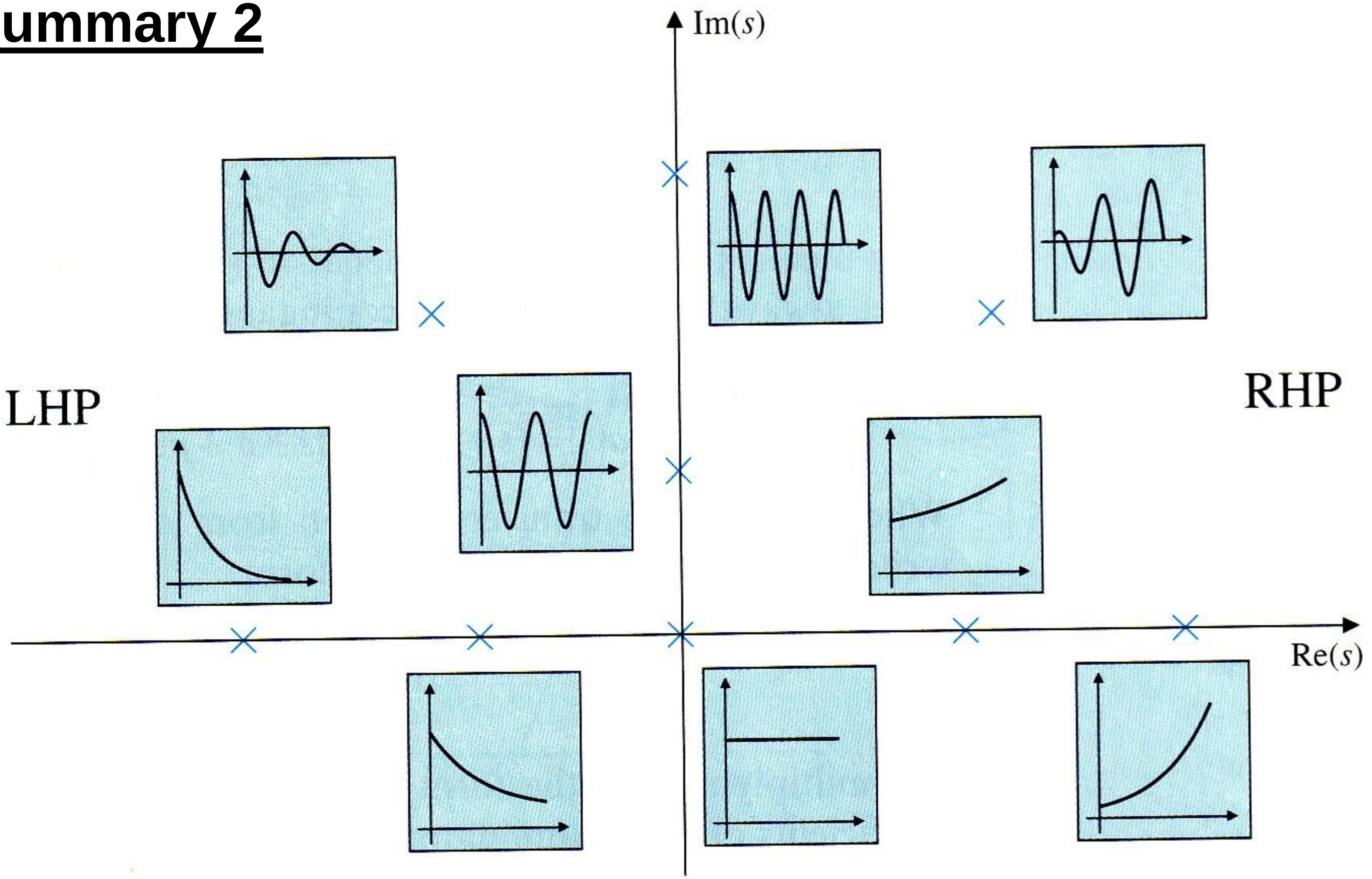


Figure: Time functions associated with points in the S-plane (LHP, left half-plane; RHP, right half-plane)

3.1. The Routh-Hurwitz stability criterion

- **The procedures for determining stability do not require finding the roots of the denominator polynomial, which can be a daunting task for a high-order system**

- **The Routh-Hurwitz stability test is a method of determining stability using simple algebraic operations on the polynomial coefficients. It is best demonstrated through an example.**

- The roots of the characteristic equations of a system may not be well easily obtained if it is in the form

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \text{----} + a_1 s + a_o, \; n>3$$

<u>Routh-Hurwitz Criterion</u>

1. Inspect the coefficients
   - If all are +ve and none of them is zero then the system can be "*Stable*".
   - If any coefficient is negative then the system is definitely "*Unstable*".
   - If any coefficient is zero then the system is most probably "*Critically Stable*".

- Examples:
  - $s^3+2s^2+3s+1 \rightarrow$ can be stable
  - $s^3-2s^2+3s+1 \rightarrow$ unstable
  - $s^3+2s^2+3s \rightarrow$ at the best, critically stable
- For those that could be stable, the coefficients are written in order called *Routh Array*.

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \ldots\ldots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots\ldots \end{array}$$

- Further rows are calculated from elements in the two rows immediately above. Successive rows are calculated until only zeros appear.

$$
\begin{array}{c|llll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} \\
s^{n-2} & b_1 & b_2 & b_3 & \ldots \\
s^{n-3} & c_1 & c_2 & c_3 & \ldots. \\
. & . & & & \\
. & . & & & \\
s^2 & x_1 & x_2 & x_3 & \\
s^1 & y_1 & y_2 & & \\
s^0 & z_1 & & &
\end{array}
$$

$$b_1 = a_{n-2}\left(\frac{a_n}{a_{n-1}}\right)a_{n-3} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}}$$

$$b_2 = a_{n-4}\left(\frac{a_n}{a_{n-1}}\right)a_{n-5} = \frac{(a_{n-1})(a_{n-4}) - a_n(a_{n-5})}{a_{n-1}}$$

$$c_1 = a_{n-3} - \left(\frac{a_{n-1}}{b_1}\right)b_2; c_2 = a_{n-5} - \left(\frac{a_{n-1}}{b_1}\right)b_3$$

- ***The number of polynomial roots in the right half plane is equal to the number of sign changes in the first column of the array***.

- Example:

$$P(s) = s^3 + s^2 + 2s + 8 = (s+2)(s^2 - s + 4)$$

The Routh array is:

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 8 \\ s^1 & -6 & \\ s^0 & 8 & \end{array}$$

- Since there are two sign changes on the first column, there are two roots of the polynomial in the right half plane: system is unstable.

- **Note:** ***The Routh-Hurwitz criterion shows only the stability of the system, it does not give the locations of the roots, therefore no information about the transient response of a stable system is derived from the R-H criterion. Also it gives no information about the steady state response. Obviously other analysis techniques in addition to the R-H criterion are needed***.

Special cases

Difficulty 1:

- When the 1st term in any row of the Routh-array is zero while rest of the row has at least one non-zero term.

Methods

1. Substitute a small +ve number ε for the zero & proceed to evaluate the rest of the Routh-array. Then examine the signs of the 1st column by letting $\varepsilon \rightarrow 0$.
2. Modify the original characteristic equation by replacing s by 1/z. Apply the Routh's test on the modified equation interms of z. The number of z-roots with +ve real parts are the same as the number of s-roots with +ve real parts. This method works in most but not all cases.

Example: $P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

$$
\begin{array}{llll}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & \varepsilon & 6 & \\
s^2 & \dfrac{4\varepsilon - 12}{\varepsilon} & 10 & \\
s^1 & \dfrac{-10\varepsilon^2 - 24\varepsilon - 72}{4\varepsilon - 12} & & \\
s^0 & 10 & &
\end{array}
$$

When we calculate the elements :

$b_1 = 0$, $b_2 = 6$, therefore we put $b_1 = \varepsilon$

and calculate the other coefficients. You should verify the results.

There are 2 sign changes regardless of ε is positive or negative. Therefore the system is unstable.

- Case 3: All elements in a row are zero.
- Example: $P(s) = s^2 + 1$

$$\begin{array}{lll} s^2 & 1 & 1 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

- Here the array cannot be completed because of the zero element in the first column.
- Another example: $P(s) = s^3 + s^2 + 2s + 2$

The array is:

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

- Example:

$$P(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

The Routh array is :

$$\begin{array}{llll} s^4 & 1 & 3 & 2 \\ s^3 & 1 & 2 & \\ s^2 & 1 & 2 & \\ s^1 & 0 & & \\ s^0 & & & \end{array}$$

Since the s^1 row contains zeros, the auxiliary polynomial is obtained from the s^2 row :

$$P_{aux}(s) = s^2 + 2$$

The derivative is $2s$, therefore 2 replaces 0 in the s^1 row, and the Routh array is then completed.

Problem: Determine the stability of the system given below
$S^5+s^4+2s^3+2s^2+3s+5=0$

Difficulty 1

- When the 1st term in any row of the routh array is zero while rest of the row has at least one non zero term.

Methods

1) substirute a small +ve number E for the zero & proceed to evaluate the rest of the Routh array then exam the sugns of the 1st column by letting E - 0
2) Modity the riginal characteristic equation by replacing s by 1/z. Apply the Routh s test on the dified equation interms of z. The no of z- roots with +ve real parts are the same as the no of s- roots with +ve real parts this method works in most but not all cases

1st Method

Routh array

S^5	1	2	3	0
S^4	1	2	5	0
S^3	E	-2	0	
S^2	$\frac{2E+2}{E}$	5		
S^1	$\frac{-2E+4-5E^2}{2E+2}$	0		
s^0	5			

There are two changes in sign & hence the system is unstable having two poles in the right half of s plane

2nd Method

Replace s by 1/Z by in the above example and rearranging

$$5*Z^5+3*Z^4+2*Z^3+2*Z^2+Z+1$$

Routh array

Z^5	5	2	1
Z^4	3	2	1
Z^3	-4/3	-2/3	0
Z^2	1/2	1	
Z^1	2	0	
Z^0	1		

- Here, there are two changes in the 1st column of the Routh array, which tell is that there are two z - roots is the right half z - plane. Therefore, the no of s-roots in the right lats s-place is also two

Difficulty 2

- When all the elements in any one low of the Routh array are zero.

- This condition indicates that there are system rically located roots in the s -plane (pair of real roots with apposite sighs and/or pair of conjugate roots on the imagnaly axis and/or complex conjugate roots forming quadrates is the s -plane)

- The polynomial whose coefficients are the elements of the raw just above the row of zeros in the R - array is called an auxiliary polynomial. This polynomial gives the no and location of root pairs of the characteristic equation which are symmetrically located in the s- plane. The order of the aux polynomial is always eve.

Method

Replace the row of zeros in the R- array by a raw of coefficients of the polynomial generated by taking the 1st derivative of the aux polynomial

Ex: $S^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

Routh array

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	0
S^3	0	0		
S^2				
S^1				
S^0				

3.2. Root Locus Technique

- Possibility of unstable operation is inherent in all feedback control systems, because of the very nature of feedback systems.
- Hence, the determination of stability of a system is necessary but not sufficient.
- Therefore, we should proceed to determine the relative stability of a stable system after we make sure the absolute stability.

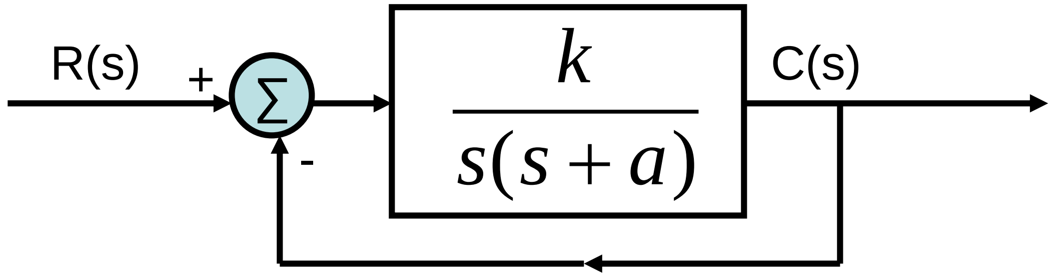
Possible solution

- Routh's criterion does not help to solve this.
- Classical technique (factorizing & determining roots of the characteristic equation) is laborious for higher degrees greater than 2.
- Root Locus technique (by W.R. Evans) is the best solution.
- The technique Provides graphical method of plotting the locus of the roots in the s-plane as a given system parameter is varied over the complete range of values (i.e. 0 to ∞)

- Roots corresponding to a particular value of the system parameter or the value of the system parameter for a desired root location can be determined from the locus. And it:
    - ✓is the best way to analyze a system's dynamic response
    - ✓also provides a measure of sensitivity of roots to the variation in the parameter
    - ✓is applicable for single as well as multiple loop systems

Root Locus

- Consider the following simple 2nd Order system.



$$\Rightarrow \frac{C(s)}{R(s)} = \frac{k}{s^2 + as + k} \longleftarrow \text{Characteristic Equation}$$

$$s_{1,2} = -\frac{a}{2} \pm \sqrt{\left[\left(\frac{a}{2}\right)^2 - k\right]} \Rightarrow \text{always stable for +ve a and k}$$

- But the dynamic behavior is controlled by the magnitudes of a & k.

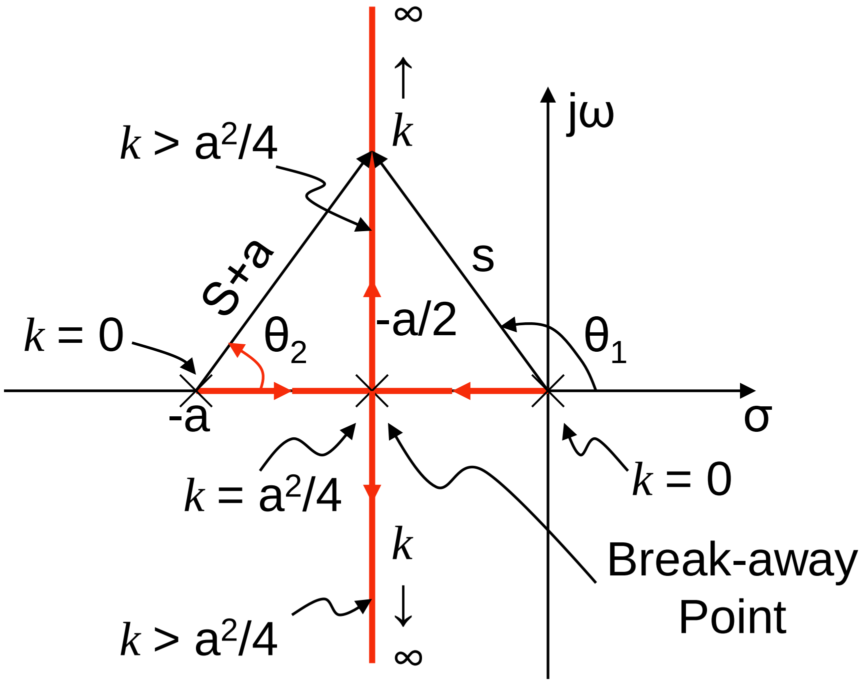
- Consider a variable gain k (common practice) while a is held constant.

   1. $0 \le k < a^2/4 \rightarrow$ real & distinct roots. Specially, When $k = 0 \rightarrow s_1 = 0$ and $s_2 = -a$ and this are called the open loop poles.

   2. $k = a^2/4 \rightarrow$ real and equal roots, $s_1 = s_2 = -a/2$.

   3. $a^2/4 < k < \infty \rightarrow$ complex conjugate roots with unvarying real part (i.e. $-a/2$)

Root Locus Plot



- Roots lying on the –ve real axis corresponds to an over-damped system, (i.e. $0 \leq k < a^2/4$).
- $k = a^2/4 \rightarrow$ critically damped system.
- $k > a^2/4 \rightarrow$ Under damped system.

- Roots move along the line $\sigma = -a/2$.
- The above locus is plotted from direct solution of the characteristic equation and is highly tedious for higher order greater than 2.

Evans simplified system

- Characteristic equation $\rightarrow 1 + P(s) = 0$
  - where $P(s) = G(s)H(s)$
  - $G(s)H(s)$ is termed as *Open loop Transfer Function* (in block diagram terminology) and *Loop Transmittance* (in signal flow graph terminology)
- $1 + P(s) = 0 \rightarrow P(s) = -1$

$$\Rightarrow \quad |P(s)| = 1 \text{ and}$$

$$\angle P(s) = \pm 180^{o}(2q+1); q = 0,1,2,...$$

- These are the two Evan's Conditions.
- A plot of the points in the complex plane that satisfy this angle criterion is the <u>Root Locus</u>. And a <u>gain</u> corresponding to a root (point on the root locus) can be determined from the magnitude criterion.

Construction of Root loci

- *1 + G(s)H(s) =0*
- *G(s)H(s)* is generally known in the factored form as it is obtained by modeling the *TF* of individual components comprising the system

$$\Rightarrow 1+G(s)H(s)=1+P(s)=1+k'\frac{\prod_{i=1}^{m}\left(\tau_{z_i}s+1\right)}{\prod_{j=1}^{n}\left(\tau_{p_j}s+1\right)}=0,$$

or

$$1+P(s)=1+k\frac{\prod_{i=1}^{m}\left(s+z_i\right)}{\prod_{j=1}^{n}\left(s+p_j\right)}=0$$

Where, $\quad k = k' \prod_{j=1}^{n} p_j \Big/ \prod_{i=1}^{m} z_i = k' \prod_{i=1}^{m} \tau_{z_i} \Big/ \prod_{j=1}^{n} \tau_{p_j}$

- k and k' are the open loop gains.
- $-z_i = -1/\tau_{zi}$ $(i=1,2,3,\ldots,m)$ are the zeros of $P(s)$.
- $-p_i = -1/\tau_{pj}$ $(j=1,2,3,\ldots,n)$ are the poles of $P(s)$.
- **Note**: In all physically real systems $n \geq m$,
  - i.e. number of poles of $P(s) \geq$ number of zero of $P(s)$

- The Evans condition will then be:

Magnitude criterion: $k\dfrac{\prod_{i=1}^{m}\left|s+z_i\right|}{\prod_{j=1}^{n}\left|s+p_j\right|}=1$ and

Angle criterion: $\sum_{i=1}^{m}\angle(s+z_i)-\sum_{j=1}^{n}\angle(s+p_j)=\pm(2q+1)180^o;q=1,2,3,\ldots$

- Follow a trial and error procedure to satisfy the angle criterion for a point on the root locus.
- After determining many loci points in this manner, draw a smooth curve through these pts.

- The value of k for particular location of s_o, from the magnitude criterion.

$$\Rightarrow k = \frac{\prod_{j=1}^{n} \left| s_o + p_j \right|}{\prod_{i=1}^{m} \left| s_o + z_i \right|}$$

$$\Rightarrow k = \frac{\text{Products of phasor lengths from } s_o \text{ to open loop poles}}{\text{Products of phasor lengths from } s_o \text{ to open loop zeros}}$$

- This is a tedious procedure.
- For quick approximate sketch of root locus the following construction rule could be used.

<u>Construction Rules</u>

- Provides a guide for selection of a trial point such that more accurate root locus can be obtained by few trials.

<u>Rule 1</u>:

- The root locus is symmetrical about the real axis.
  - Proof: Roots are either real or complex conjugates or combination of both.

<u>Rule 2</u>:

- As k increases form 0 to ∞, each branch of the root locus originates from an open-loop pole with $k=0$ and terminates either on an open-loop zero or on infinity with $k=\infty$.
- The number of branches terminating at infinity equals the number of open-loop poles minus number of zeros.

<u>Rule 3</u>:

- A point on the real axis lies on the locus if the number of open-loop poles plus zeros on the real axis to the right of this point is odd.

Rule 4:

- The (n-m) branches of the root locus which tend to infinity, do so along straight line asymptotes whose angles are given by:

$$\phi_A = \frac{(2q+1)180}{n-m}; q = 0,1,2,\ldots,(n-m-1)$$

Rule 5:

- The asymptotes cross the real axis at a point known as <u>centroid</u>, determined by the relation:

$$\frac{\text{(sum of real parts of poles-sum of real parts of zeros}}{\text{(number of poles-number of zeros)}}$$

Rule 6:

- The brake away points of the root locus are the solutions of $dk/ds=0$.
  - *Defn.: Brake away points are points at which multiple roots of the characteristic equation occur.*

Note:

i. Since the characteristic equation can have real as well as complex multiple roots, its root locus can have real as well as complex break away points, but because of conjugate symmetry of the root loci, the break away points must either be on the real axis or occur in complex conjugate pairs.

ii. All the roots of equation $dk/ds=0$ are not breakaway points. The actual breakaway points are roots at which the root locus angle criterion is met.

Breakaway directions of Root locus Branches

- The root locus branches must approach or leave the breakaway point on the real axis at an angle of $\pm 180^o/r$, where r is the number of root locus branches approaching or leaving the points.

Rule 7:

- The angle of departure from an open-loop pole is given by:

$$\phi_p = \underline{+}\ 180^o(2q+1)+\phi,\ q= 0,1,2,\ \ldots$$

  - where ϕ is the net angle contribution, at this pole, of all other open-loop poles and zeros.

- Similarly, the angle of arrival at an open-loop zero is given by:

$$\phi_z = \underline{+}\ 180^o(2q+1)-\phi,\ q= 0,1,2,\ \ldots$$

  - where ϕ means the net angle contribution at the zero under consideration of all other open-loop poles & zeros.

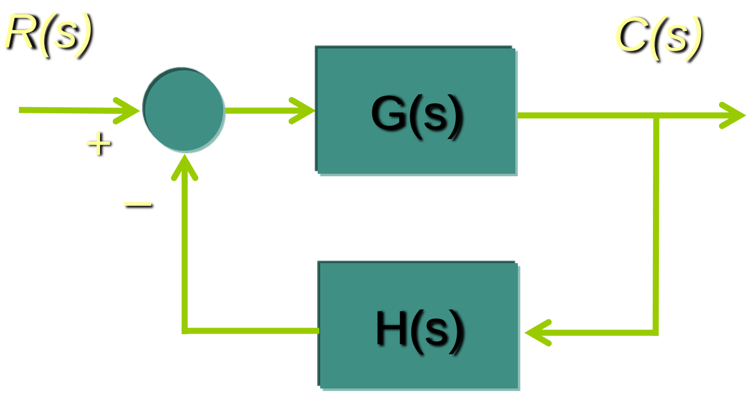
- The angle of departure from a real open-loop pole or the angle of arrival at a real open-loop zero is always 0^0 or 180^0. Hence, no need to calculate these angles for real pole and zero, but need for complex poles & zeros.

Rule 8:

- The intersection of root locus branches with the imaginary axis can be determined by use of the Routh criterion.

The General Root Locus Method

- Consider the general system



where

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+GH(s)}$$

- The characteristic equation is

$$1+GH(s)=0$$

or

$$GH(s)=-1$$

or

$$|GH(s)|=1$$

$$\angle GH(s)=(2k+1)\pi$$

$$k=0,\pm 1,\pm 2\cdots$$

The General Root Locus Method

- All values of s which satisfy

$$|GH(s)| = 1 \quad ; \quad \angle GH(s) = (2k+1)\pi \qquad k = 0, \pm 1, \pm 2 \cdots$$

are roots of the closed-loop characteristic equation.

- Consider the following general form

$$GH(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Note: p_i maybezero.

The General Root Locus Method

- Then

$$\left|GH(s)\right| = \frac{|K| \prod_{i=1}^{m} |s+z_i|}{\prod_{i=1}^{n} |s+p_i|} = 1$$

$$\angle GH(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = (2k+1)\pi$$

$$k = 0, \pm 1, \pm 2 \cdots$$

Root Locus Method: Geometric Interpretation

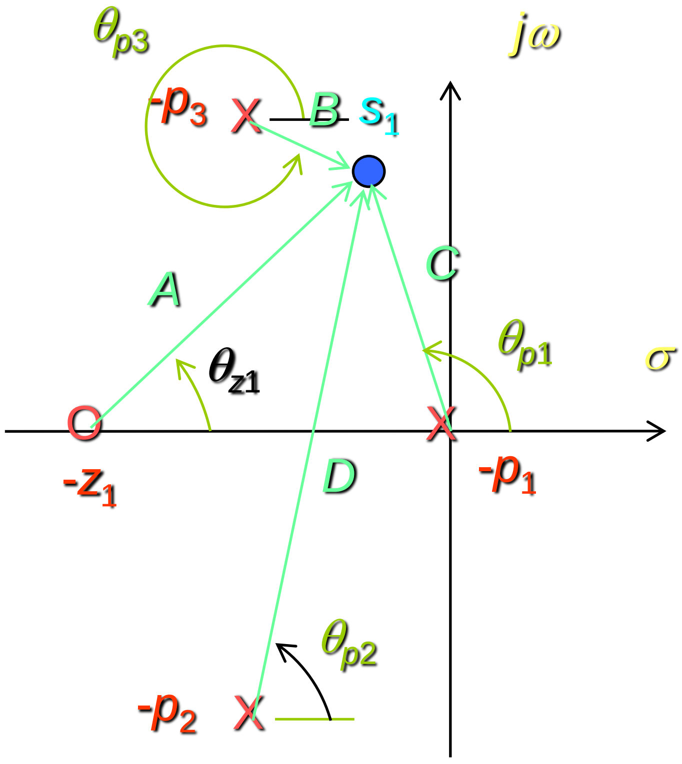
- Consider the example

$$GH(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)}$$

- Then the values of $s = s_1$ which satisfy

$$\frac{|K|\,|s+z_1|}{|s|\,|s+p_2|\,|s+p_3|} = 1$$

$$\angle(s+z_1) - (\angle s + \angle(s+p_2) + \angle(s+p_3)) = (2k+1)\pi$$



are on the loci and are roots of the characteristic equation.

Root Locus Method: Geometric Interpretation

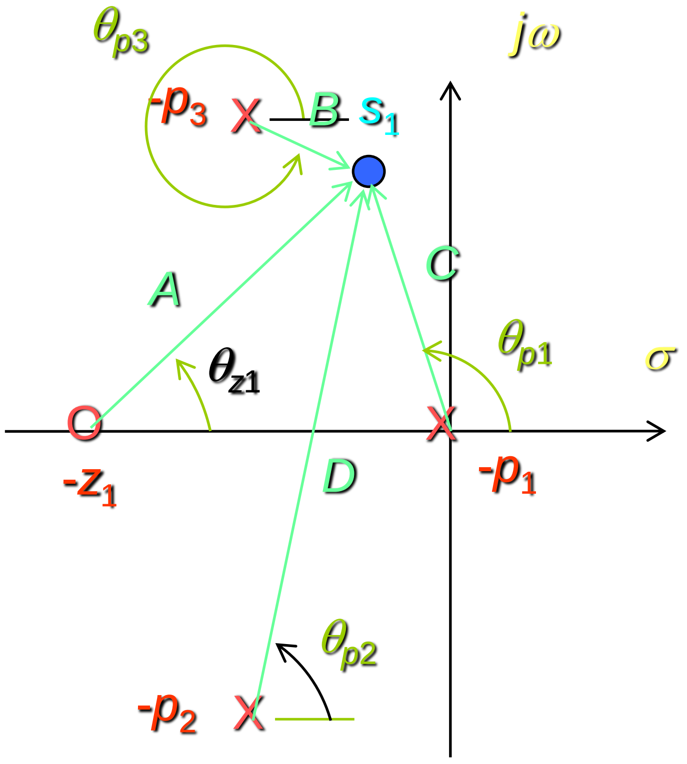
- In terms of the vectors, the condition for $s = s_1$ to be on the root loci are

$$\frac{|K|A}{BCD} = 1 \quad \text{or} \quad \frac{A}{BCD} = \frac{1}{|K|}$$

and

$$\theta_{z1} - (\theta_{p1} + \theta_{p2} + \theta_{p3}) = (2k+1)\pi$$

$$k = 0, \pm 1, \pm 2, \cdots$$



Root Locus Method

- When plotting the loci of the roots as $K = 0 \rightarrow \infty$, the magnitude condition is always satisfied.
- Therefore, a value of $s = s_1$ that satisfies the angle condition, is a point of the root loci.
- The magnitude condition may then be used to determine the gain K corresponding to that value s_1 .
- How can we easily determine if the angle condition is satisfied?

Root Locus Construction Rules

1. The loci start ($K = 0$) at the poles of the open-loop system. There are n loci .
2. The loci terminate ($K \to \infty$) at the zeroes of the open-loop system (include zeroes at infinity).

- For our example system

$$|GH(s)| = \frac{|s+z_1|}{|s|\,|s+p_2|\,|s+p_3|} = \frac{1}{|K|}$$

- Therefore, as $K \to 0$, $GH(s) \to \infty$, the poles of the loop transfer function.
- As $K \to \infty$, $GH(s) \to 0$, the zeroes of the loop transfer function.

Root Locus Construction Rules

3. The root loci are symmetrical about the real axis.
4. As $K \to \infty$ the loci approach asymptotes. There are $q = n - m$ asymptotes and they intersect the real axis at angles defined by

$$\frac{(2k+1)\pi}{q}, \quad k = 0, \pm 1, \pm 2, \cdots$$

- The roots with imaginary parts always occur in conjugate complex pairs.
- When the loci approach infinity, the angles from all the poles and zeroes are equal. The angle condition then is $m\theta - n\theta = (2k + 1)\pi$

Root Locus Construction Rules

5. The asymptotes intersection point on the real is defined by

$$\sigma_a = \frac{\sum \text{poles of } GH(s) - \sum \text{zeroes of } GH(s)}{q}$$

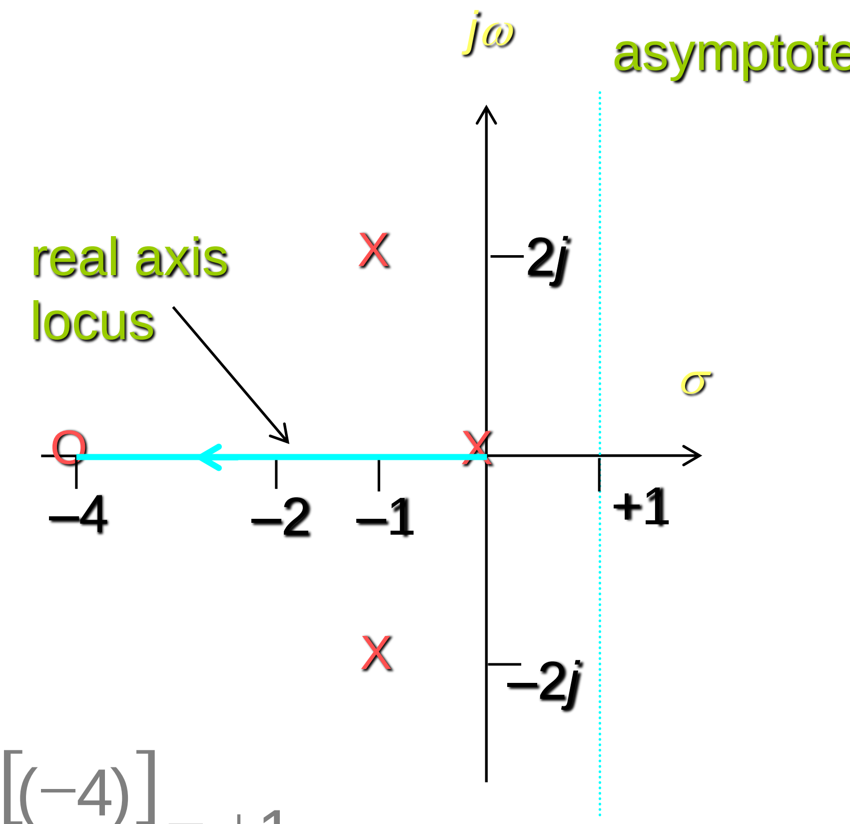
6. Real axis sections of the root loci exist only where there is an odd number of poles and zeroes to the right.

- The angles from poles and zeroes to the left of s_1 are zero. The angles from poles and zeroes to the right are $-\pi$. An odd number are required to satisfy the angle condition.

Root Locus Construction Rules Example

- Consider our example with $z_1 = 4$,$p_{12} = 1 \pm 2j$

$$GH(s) = \frac{K(s+4)}{s(s+1+2j)(s+1-2j)}$$



- Asymptotes:

$$\text{angles} = \frac{(2k+1)\pi}{3-1} = \pm\frac{\pi}{2}$$

$$\sigma_a = \frac{\left[-0-(1+2j)-(1-2j)\right]-\left[(-4)\right]}{3-1} = +1$$

Root Locus Construction Rules

7. The angles of departure, θ_d from poles and arrival, θ_a to zeroes may be found by applying the angle condition to a point very near the pole or zero.

- The angle of arrival at the zero, $-z_1$ is obtained from

$$\theta_{az1} + \sum_{i=2}^{m} \angle(-z_1 + z_i) - \sum_{i=1}^{n} \angle(-z_1 + p_i) = (2k+1)\pi$$

Root Locus Construction Rules Example

- Departure angle from p_2 .

  $\theta_{z1} = \tan^{-1}(2/3) = 33.7^\circ$

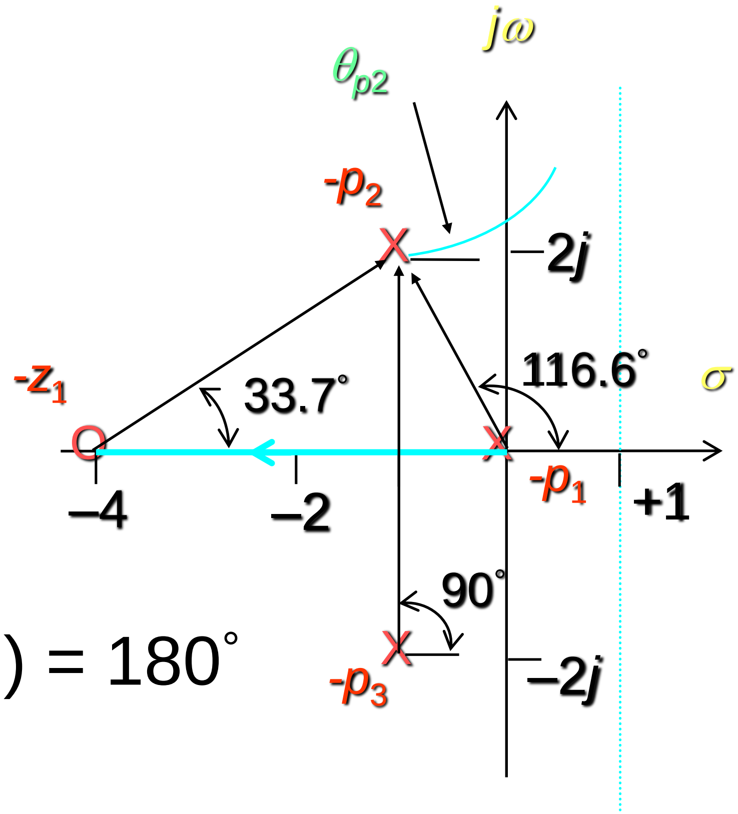
  $\theta_{p1} = \tan^{-1}(-2/1) = 116.6^\circ$

  $\theta_{p3} = 90^\circ$

- Then

  $33.7^\circ - (90^\circ + 116.6^\circ + \theta_{p2}) = 180^\circ$

  $\theta_{p2} = -352.9^\circ = +7.1^\circ$



Root Locus Construction Rules

8. The imaginary axis crossing is obtained by applying the Routh-Hurwitz criterion and checking for the gain that results in marginal stability. The imaginary components are found from the solution of the resulting auxiliary equation.

- Marginal stability refers to the point where the roots of the closed-loop system are on the stability boundary, i.e. the imaginary axis.

Root Locus Construction Rules Example

- Imaginary axis crossing:

  Characteristic equation

$$s(s+1+2j)(s+1-2j)+K(s+4)=0$$

$$s^3+2s^2+(5+K)s+4K=0$$

Routh table

s^3	1	$5+K$	0
s^2	2	$4K$	0
s	$5-K$	0	0
s^0	$4K$	0	

- For marginal stability, K = 5 and the auxiliary equation is

$$2s^2+20=0$$

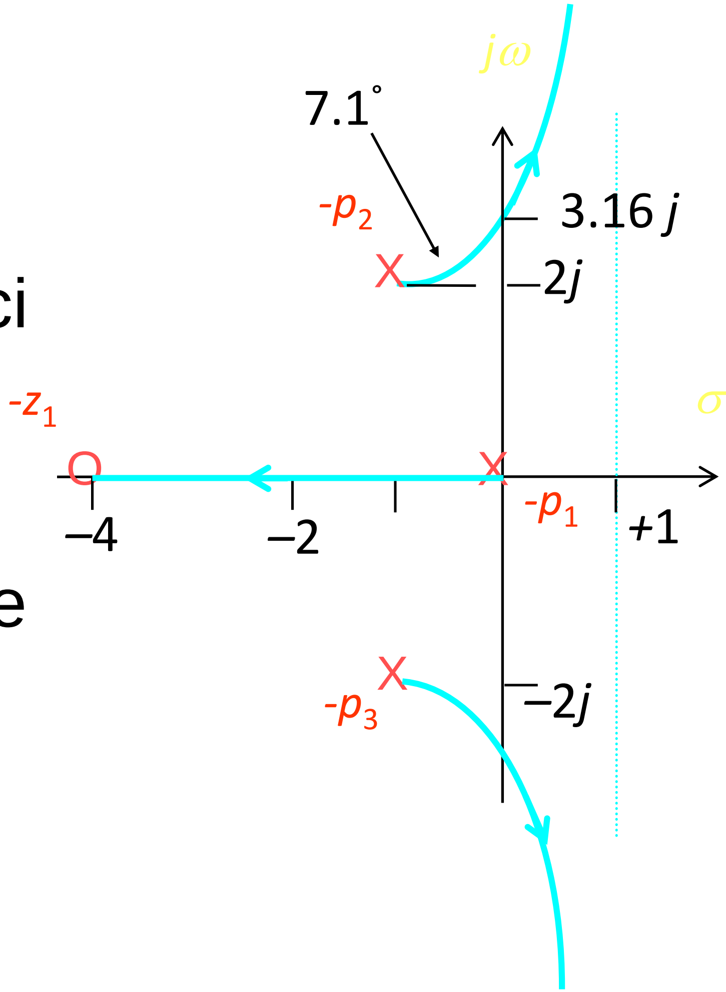
$$s=\pm\sqrt{10}j=\pm 3.16j$$

- Therefore, the imaginary axis intersection is $\pm 3.16j$

Root Locus Construction Rules Example

- Summary:

  There are three root loci. One on the real axis from $-p_1$ to $-z_1$, and a pair of loci from $-p_2$ and $-p_3$ to zeroes at infinity along the asymptotes. The departure angle from these poles is $\pm\, 7.1^{\circ}$ and an imaginary axis crossing at $s = \pm 3.16j$.

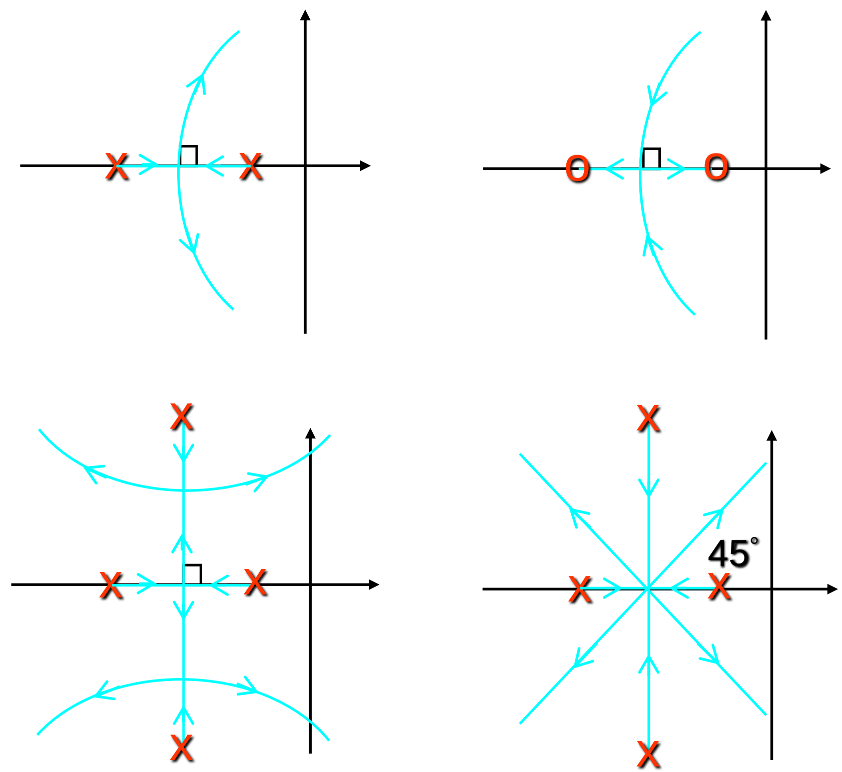


Root Locus Construction Rules

- Breakaway Points:

  When two or more loci meet, they will breakaway from this point at particular angles. The point is known as a breakaway point. It corresponds to multiple roots.

Some examples



Root Locus Construction Rules

9. The angle of breakaway is 180°/*k* where *k* is the number of converging loci.
   The location of the breakaway point is found from

$$\frac{dK}{ds} = 0 \quad \text{or} \quad \frac{d[GH(s)]}{ds} = 0$$

- Note: $K = -[GH(s)]^{-1}$

$$\frac{dK}{ds} = [GH(s)]^{-2} \frac{d[GH(s)]}{ds} = 0$$

- Also,

$$\frac{d[GH(s)]}{ds} = \frac{d[N(s)/D(s)]}{ds}$$

$$= \frac{N'(s)}{D(s)} - \frac{N(s)D'(s)}{D(s)^2} = 0$$

$$\boxed{D(s)N'(s) - N(s)D'(s) = 0}$$

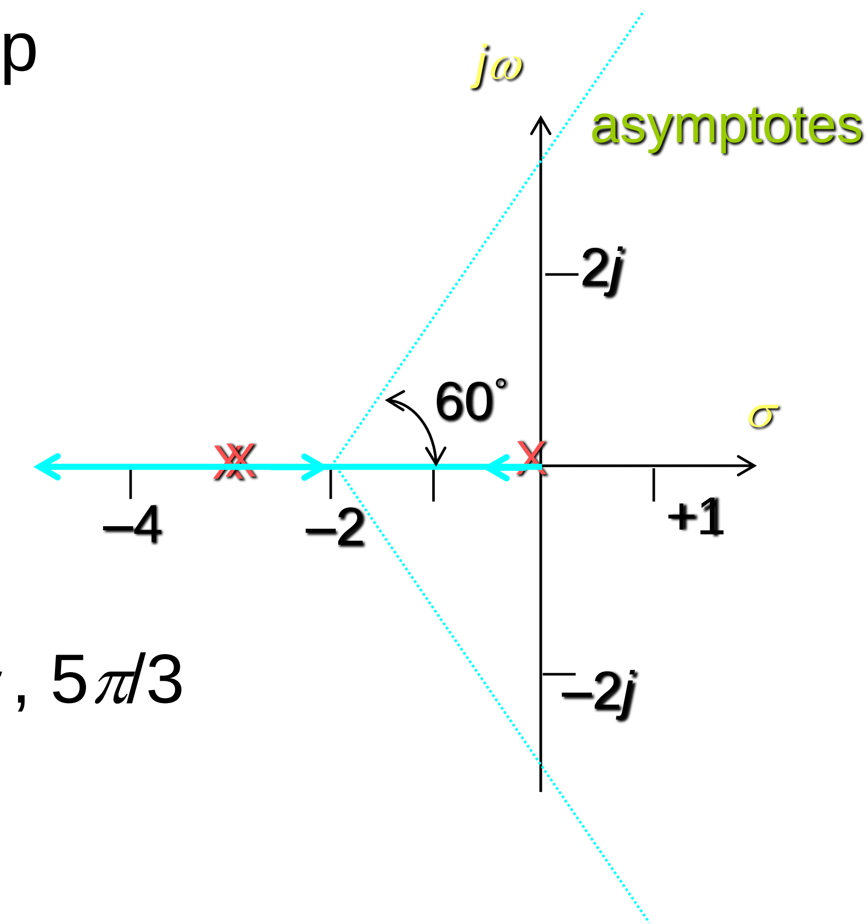
Root Locus Plot: Breakaway Point Example

- Consider the following loop transfer function.

$$GH(s) = \frac{K}{s(s+3)^2}$$

- Real axis loci exist for the full negative axis.
- Asymptotes:
  $\text{angles} = \frac{(2k+1)\pi}{3} = \pi/3 \, , \; \pi \, , \; 5\pi/3$

$$\sigma_a = \frac{(-3-3-0)-(0)}{3} = -2$$



Root Locus Plot: Breakaway Point Example

- Determine the breakaway points from

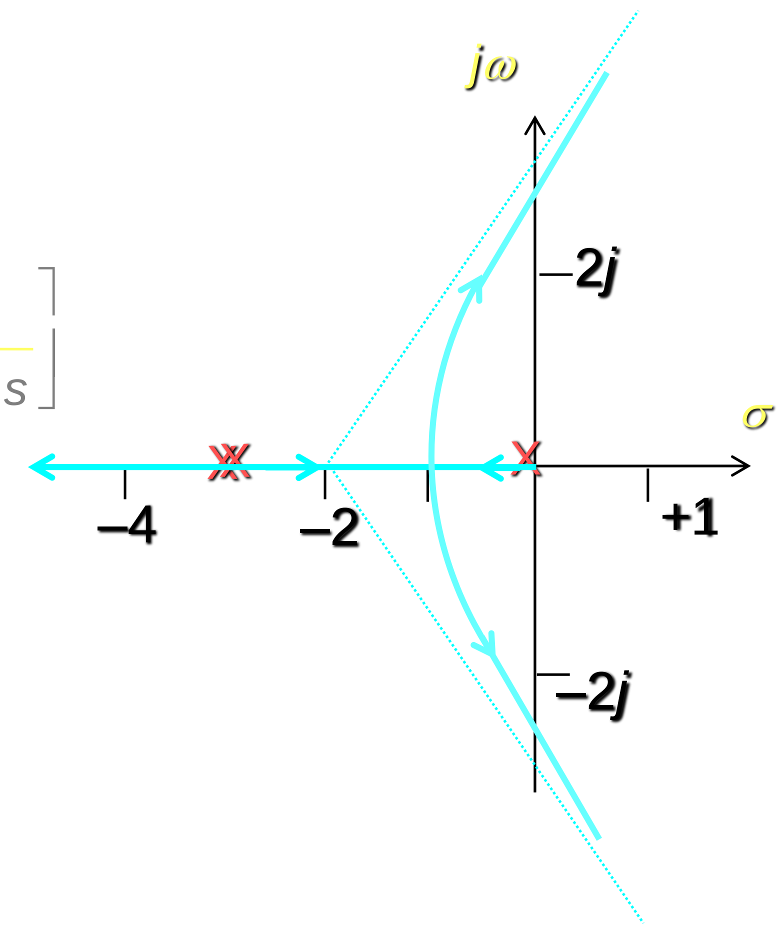
$$\frac{d}{ds}\left[\frac{K}{s(s+3)^2}\right]=\frac{d}{ds}\left[\frac{K}{s^3+6s^2+9s}\right]$$

$$=\frac{-K(3s^2+12s+9)}{(s^3+6s^2+9s)^2}=0$$

then

$$s^2+4s+3=(s+1)(s+3)=0$$

$$s=-1,-3$$

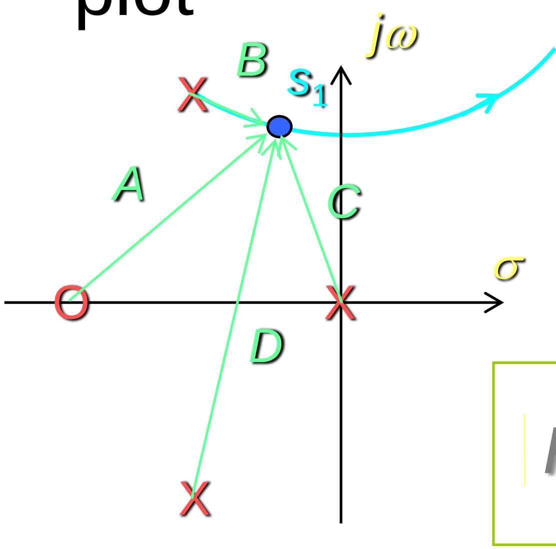


Root Locus Construction Rules

10. For a point on the root locus, $s = s_1$ calculate the gain, K from

$$|K| = \frac{|s_1 + p_1|\,|s_1 + p_2|\,|\quad|\cdots}{|s_1 + z_1|\,|s_1 + z_2|\,|\quad|\cdots}$$

- Alternately, K may be determined graphically from the root locus plot



$$|K| = \frac{BCD}{A}$$

Summary of Root Locus Plot Construction

- Plot the poles and zeros of the open-loop system.
- Find the section of the loci on the real axis (odd number of poles an zeroes to the right).
- Determine the asymptote angles and intercepts.

$$\text{angles} = \frac{(2k+1)\pi}{q}, \quad q = n - m, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\sigma_a = \frac{\sum \text{poles} - \sum \text{zeroes}}{q}$$

Summary of Root Locus Plot Construction

- Determine departure angles. For a pole $-p_1$

$$\angle(-p_1+z_1)+\angle(-p_1+z_2)+\ldots-\theta_{p1}-\angle(-p_1+p_2)-\ldots=(2k+1)\pi$$

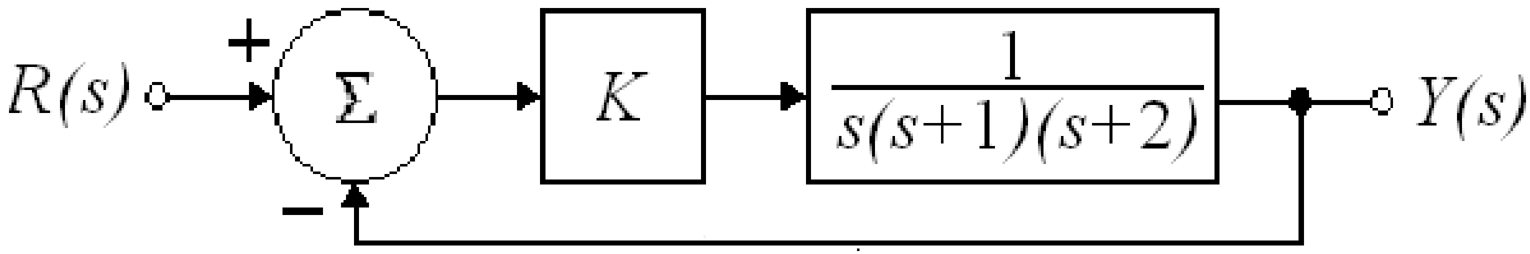
- Check for imaginary axis crossings using the Routh-Hurwitz criterion.
- Determine breakaway points.

$$\text{angle} = \pi/k \ , \ k = \#\text{ of converging loci}$$

$$\text{location from } \frac{d[GH(s)]}{ds} = 0$$

- Complete the plot.

1Q.) Sketch the root locus of the following system:



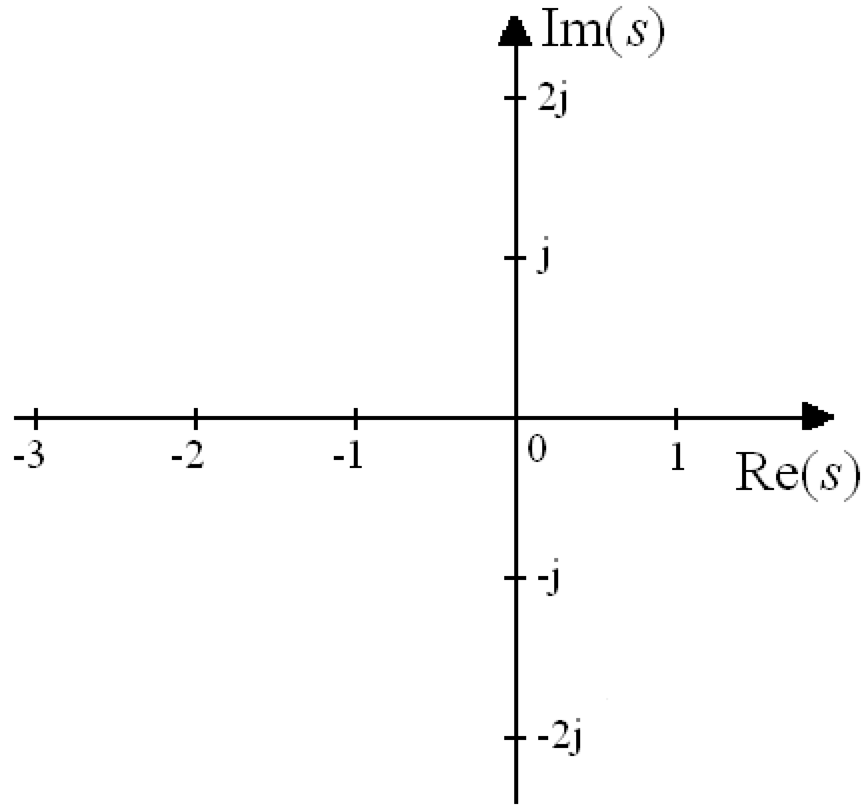
Draw the *n* poles and *m* zeros of *G*(*s*)*H*(*s*) using x and o respectively.

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

- 3 poles:

  $p_1 = 0$; $p_2 = -1$; $p_3 = -2$

  No zeros



Applying Step #1

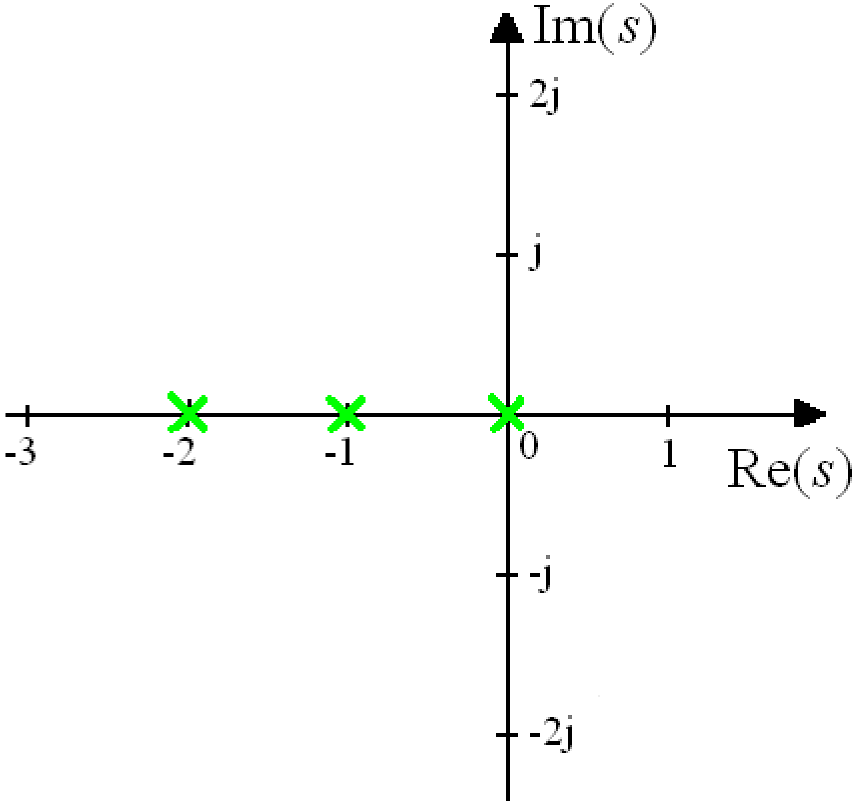
Draw the *n* poles and *m* zeros of $G(s)H(s)$ using x and o respectively.

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

- 3 poles:

  $p_1 = 0$; $p_2 = -1$; $p_3 = -2$

- No zeros



Rule #2

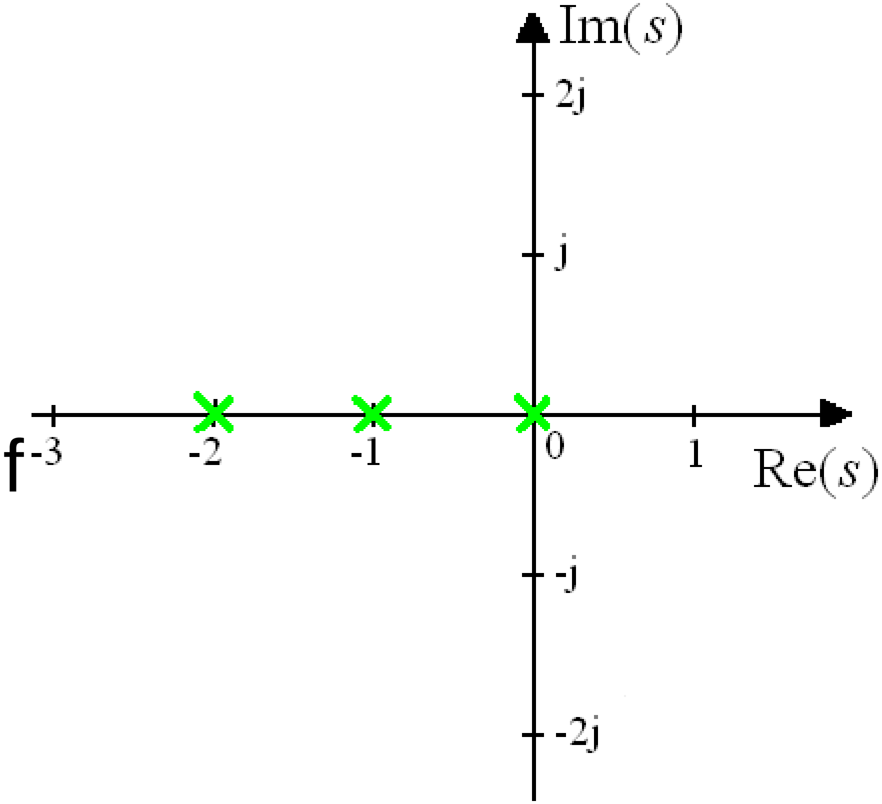
- The **loci on the real axis** are to the **left of an ODD number of REAL poles and REAL zeros** of $G(s)H(s)$

Second step: Determine the loci on the real axis. Choose a arbitrary test point. If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus

Applying Step #2

Determine the loci on the real axis:

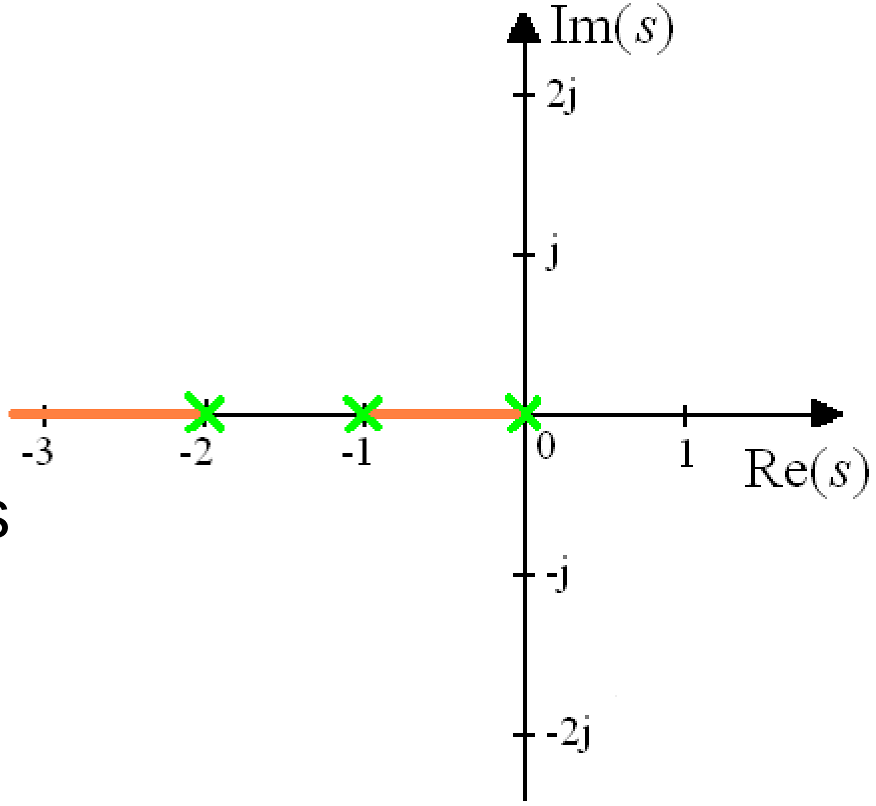
- Choose a arbitrary test point.
- If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus



Applying Step #2

Determine the loci on the real axis:

- Choose a arbitrary test point.
- If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus



Rule #3

<u>Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):</u>

- The **root loci for very large values of s** must be **asymptotic** to **straight lines originate on the real axis at point:**

$$s = \alpha = \frac{\sum_n p_i - \sum_m z_i}{n - m} \qquad \phi_l = \frac{\pm 180^o \left(2l + 1\right)}{n - m}$$

radiating out from this point at angles:

<u>Third step</u>: Determine the *n - m* asymptotes of the root loci. Locate s = α on the real axis. Compute and draw angles. Draw the asymptotes using dash lines.

Applying Step #3

Determine the *n - m* asymptotes:

- Locate s = α on the real axis:

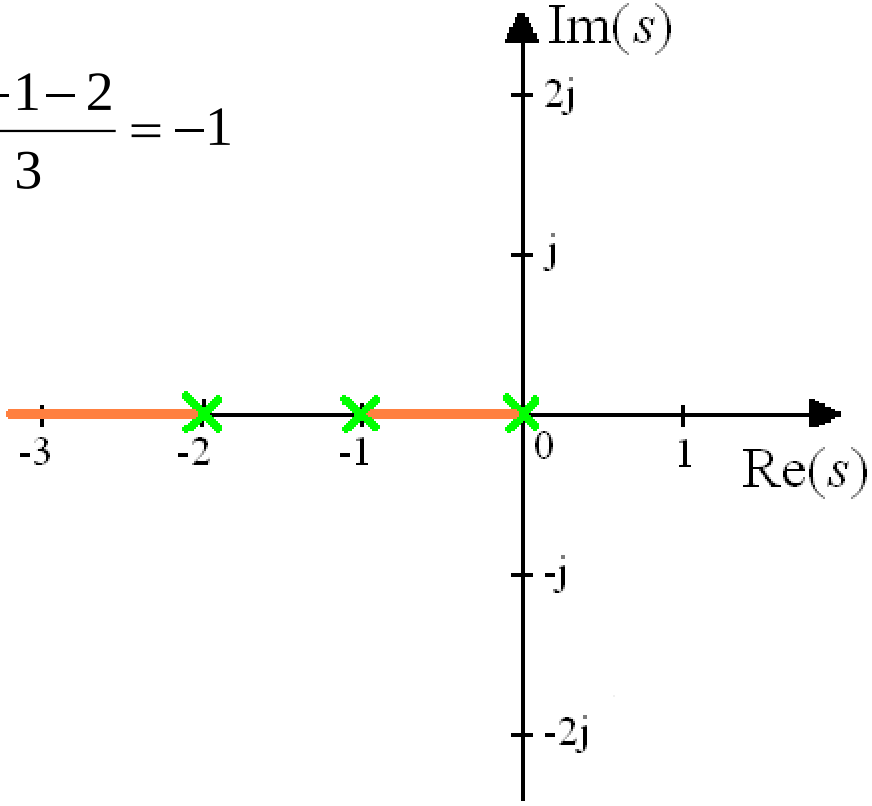
$$s = \alpha = \frac{p_1 + p_2 + p_3}{3-0} = \frac{0-1-2}{3} = -1$$

- Compute and draw angles:

$$\phi_l = \frac{\pm 180(2l+1)}{n-m} \qquad l = 0,1,2,\dots$$

$$\Rightarrow \begin{cases} \phi_0 = \dfrac{\pm 180^0(2\times 0+1)}{3-0} = \pm 60^0 \\ \phi_1 = \dfrac{\pm 180^0(2\times 1+1)}{3-0} = \pm 180^0 \end{cases}$$

- Draw the asymptotes using dash lines.



Applying Step #3

Determine the *n - m* asymptotes:

- Locate s = α on the real axis:

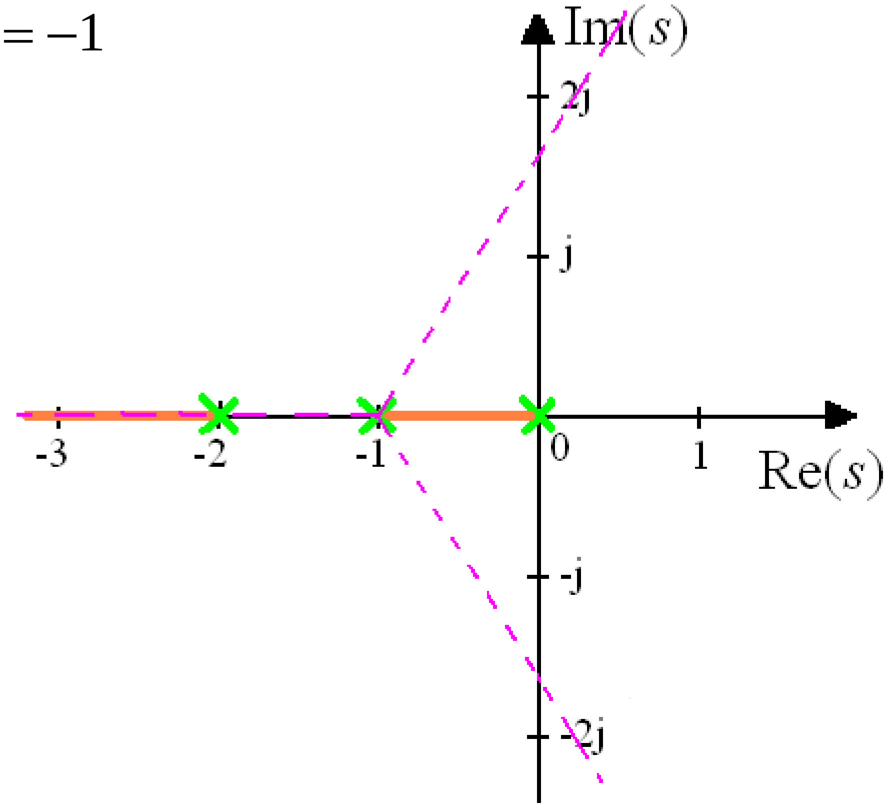
$$s = \alpha = \frac{p_1 + p_2 + p_3}{3-0} = \frac{0-1-2}{3} = -1$$

- Compute and draw angles:

$$\phi_l = \frac{\pm 180(2l+1)}{n-m} \qquad l = 0, 1, 2, \ldots$$

$$\Rightarrow \begin{cases} \phi_0 = \dfrac{\pm 180^0(2\times 0+1)}{3-0} = \pm 60^0 \\ \phi_1 = \dfrac{\pm 180^0(2\times 1+1)}{3-0} = \pm 180^0 \end{cases}$$

- Draw the asymptotes using dash lines.



Breakpoint Definition

- The breakpoints are the points in the s-domain where **multiples** roots of the characteristic equation of the feedback control occur.

- These points correspond to intersection points on the root locus.

Rule #4

Given the characteristic equation is $KG(s)H(s) = -1$

- The **breakpoints are the closed-loop poles that satisfy:**

$$\frac{dK}{ds} = 0$$

Fourth step: Find the breakpoints. Express K such as:

$$K = \frac{-1}{G(s)H(s)}.$$

Set $dK/ds = 0$ and solve for the poles.

Applying Step #4

Find the breakpoints.

- Express K such as:

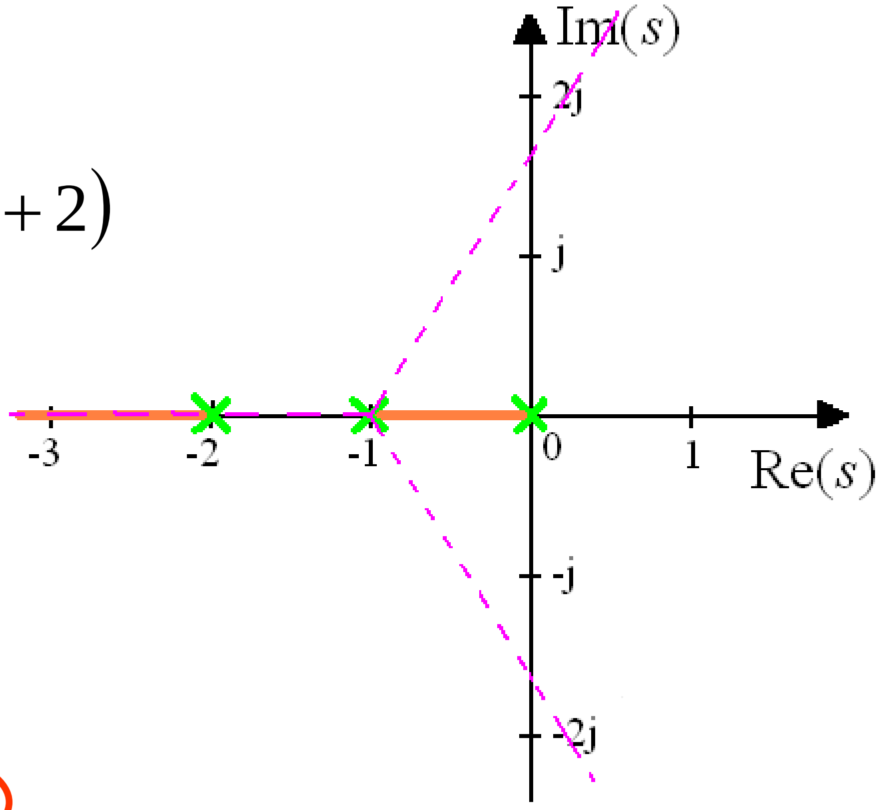
$$K = \frac{-1}{G(s)H(s)} = -s(s+1)(s+2)$$

$$K = -s^3 - 3s^2 - 2s$$

Set $dK/ds = 0$ and solve for the poles.

$$-3s^2 - 6s - 2 = 0$$

$$\cancel{s_1 = -1.5774}, \quad s_2 = -0.4226$$



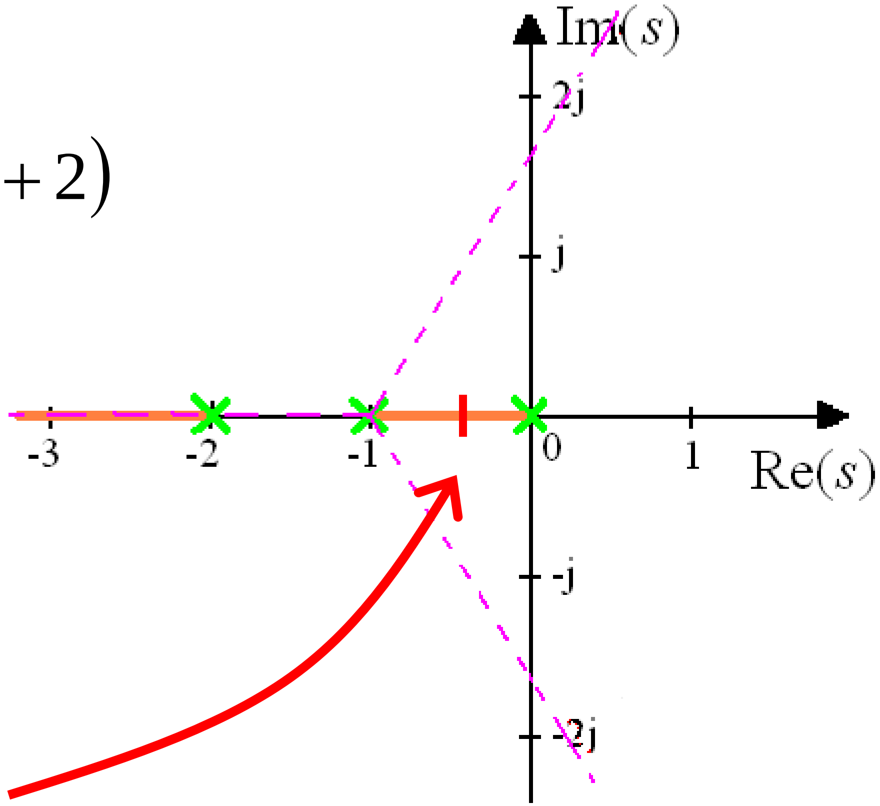
Applying Step #4

Find the breakpoints.

- Express K such as:

$$K = \frac{-1}{G(s)H(s)} = -s(s+1)(s+2)$$

$$K = -s^3 - 3s^2 - 2s$$



- Set dK/ds = 0 and solve for the poles.

$$-3s^2 - 6s - 2 = 0$$

$$\cancel{s_1 = -1.5774}, \quad s_2 = -0.4226$$

Recall Rule #1

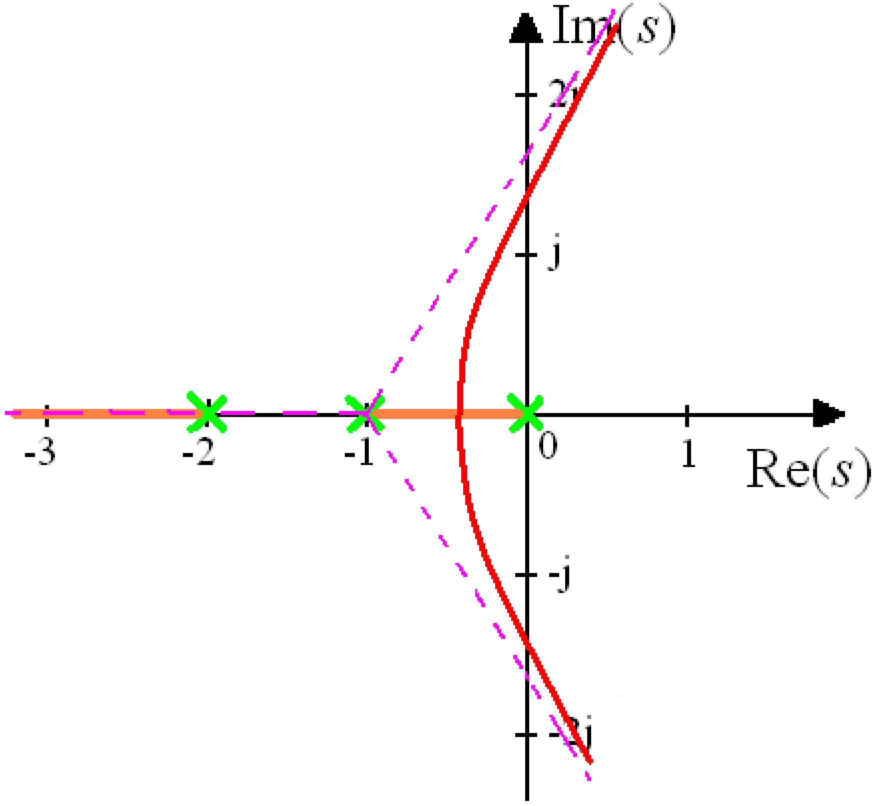
Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):

- The ***n* branches** of the root locus **start at the *n* poles.**
- ***m*** of these *n* branches **end on the *m* zeros**
- The ***n-m* other branches terminate at infinity** along asymptotes.

Last step: Draw the *n-m* branches that terminate at infinity along asymptotes

Applying Last Step

Draw the *n-m* branches that terminate at infinity along asymptotes



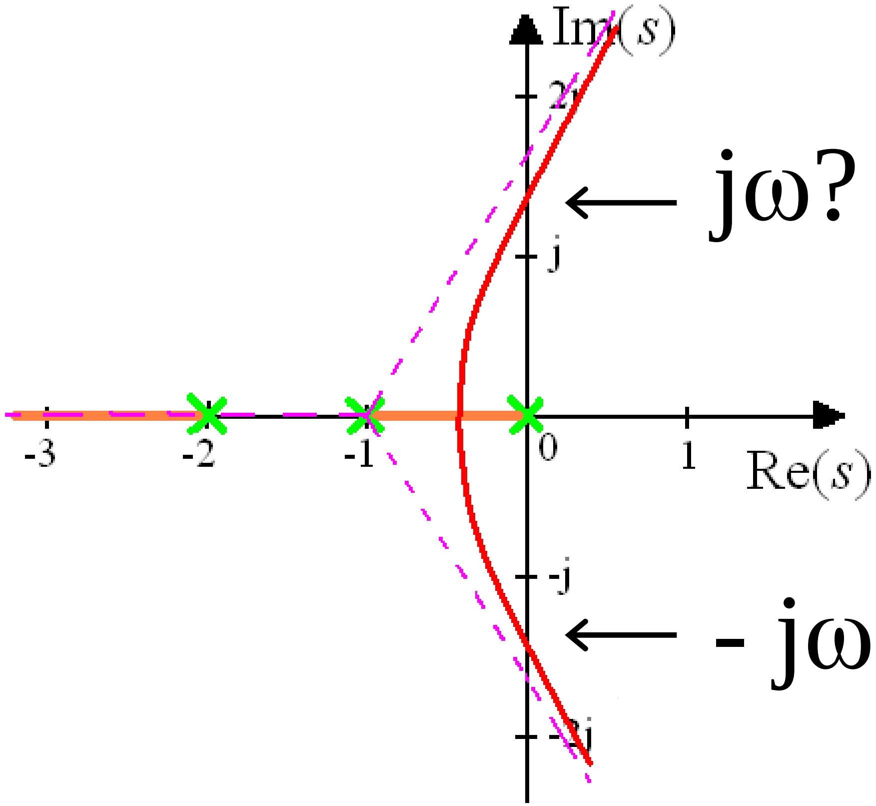
Points on both root locus & imaginary axis?

- Points on imaginary axis satisfy:

$$s = j\omega$$

- Points on root locus satisfy:

$$1 + KG(s)H(s) = 0$$



- Substitute *s=jω* into the characteristic equation and solve for *ω*. $\omega = 0 \;\; or \;\; \omega = \pm\sqrt{2}$

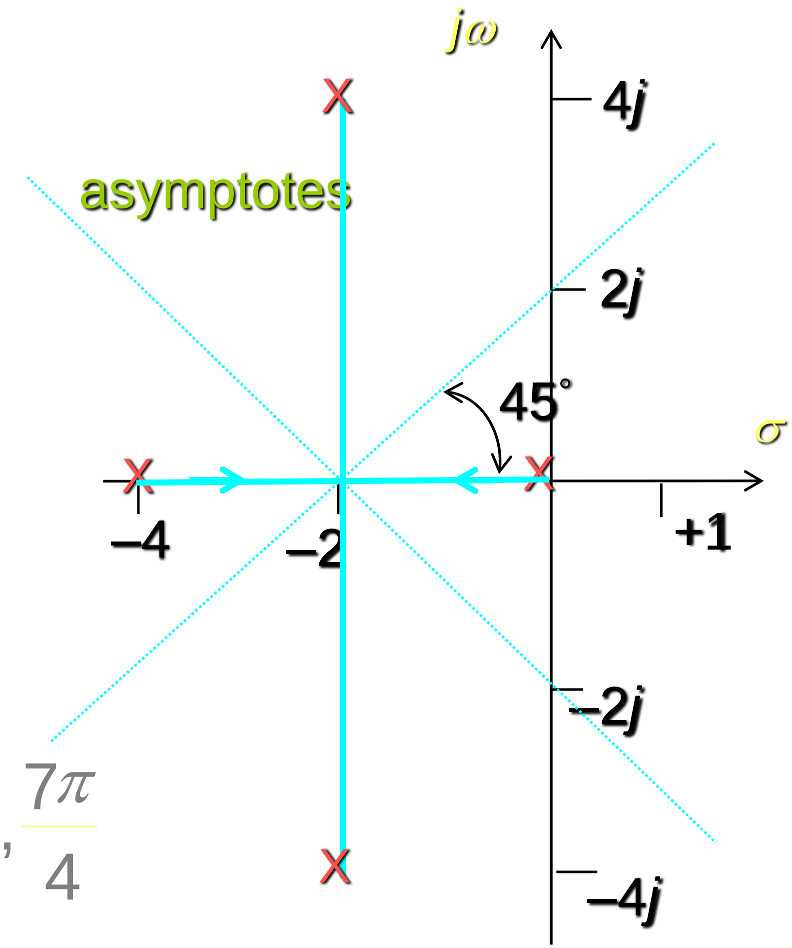
2Q.) Draw the Root Locus for the system equation given below:

- Loop Transfer function:

$$GH(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

- Roots:
  s = 0, s = –4, s = –2 ± 4*j*
- Real axis segments: between 0 and –4 .
- Asymptotes:
  angles = $\frac{(2k+1)\pi}{4-0} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\sigma_a = \frac{(-4-2-2-0)}{4} = -2$$



Continued..

- Breakaway points:

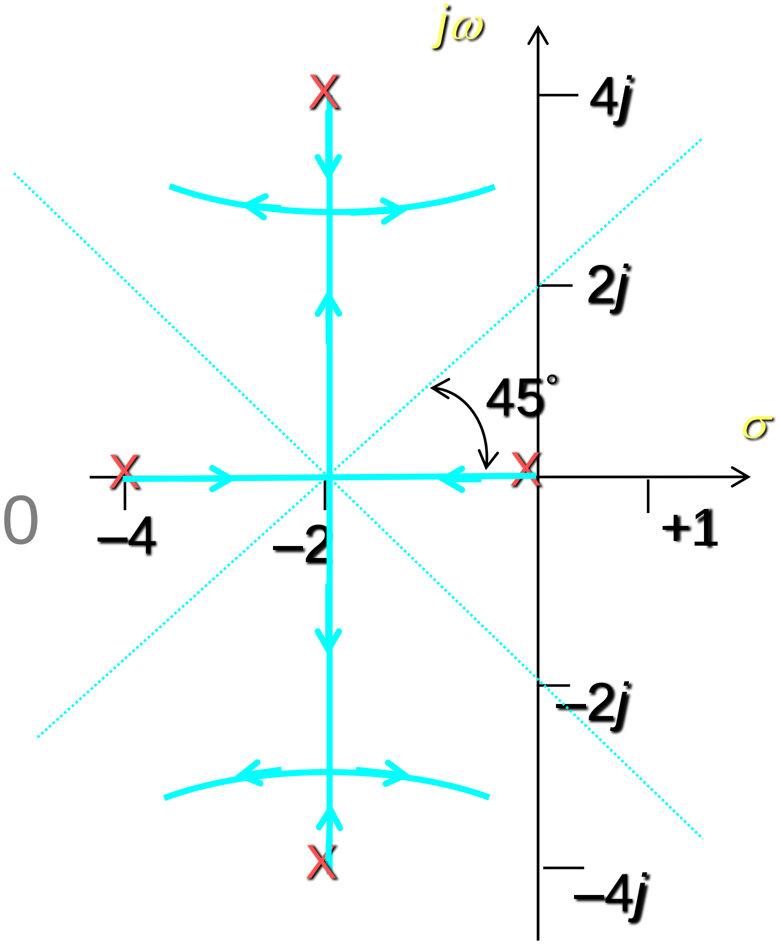
$$\frac{d}{ds}\left[\frac{K}{s^4+8s^3+36s^2+80s}\right]$$

$$=-\frac{K(4s^3+24s^2+72s+80)}{(s^4+8s^3+36s^2+80s)^2}=0$$

$$\text{or}\quad s^3+6s^2+18s+20=0$$

solving, $s_b=-2,\ -2\pm 2.45j$

- Three points that breakaway at 90˚ .



Continued..

- The imaginary axis crossings:

  Characteristic eqn.

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

Routh table

s^4	1	36	K
s^3	8	80	0
s^2	26	K	0
s	$2080-8K/26$	0	0
s^0	K	0	0

- Condition for critical stability

  $2080-8K/26 > 0$ or

  $K<260$

  The auxiliary equation

  $26\,s^2 + 260 = 0$

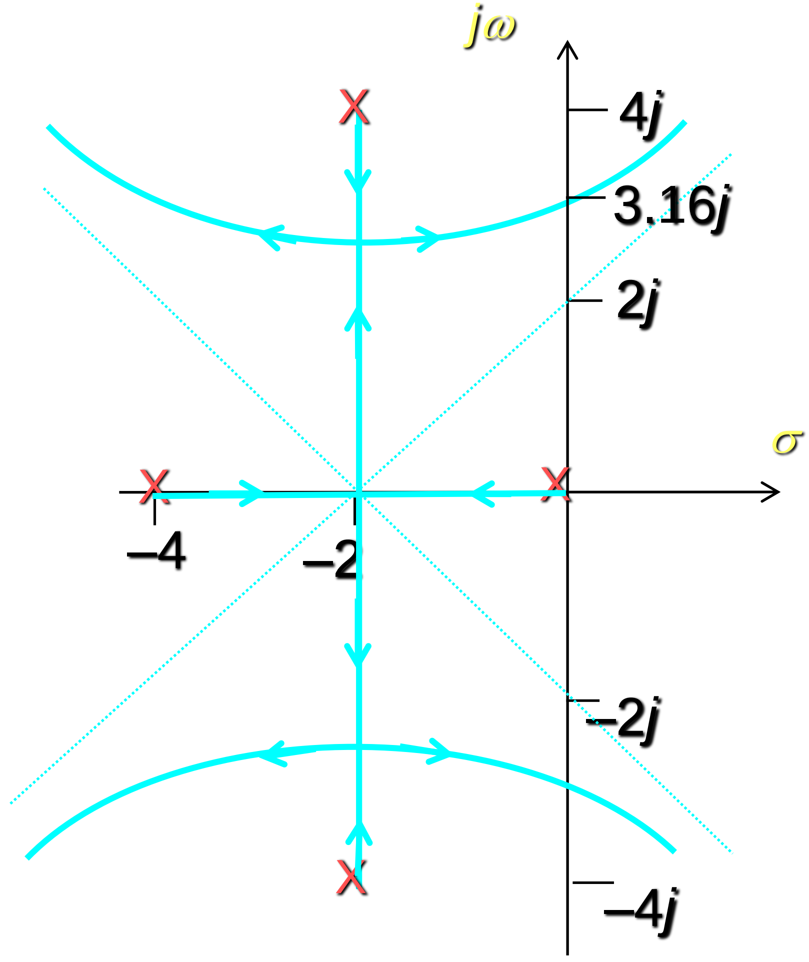
  solving

$$s = \pm\sqrt{10}j = \pm 3.16j$$

Continued..

- The final plot is shown on the right.

- What is the value of the gain K corresponding to the breakaway point at $s_b = -2 \pm 2.45j$?



Continued..
Gain Calculations

- From the general magnitude condition the gain corresponding to the point s_1 on the loci is

$$K = \prod_{i=1}^{n} |s_1 + p_i| \Big/ \prod_{i=1}^{m} |s_1 + z_i|$$

- For the point $s_1 = -2 + 2.45j$

$$K = |-2 + 2.45j| \cdot |-2 + 2.45j + 4| \cdot |-2 + 2.45j + 2 + 4j| \cdot |-2 + 2.45j + 2 - 4j| \,/\, 1.0$$

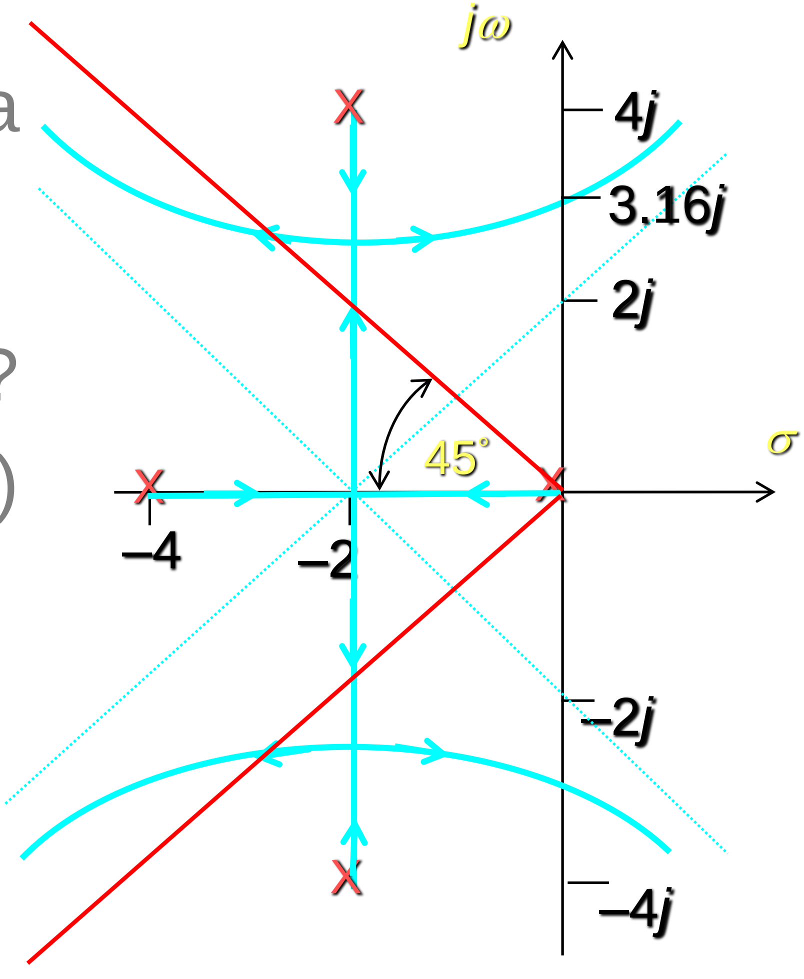
$$= 3.163 \cdot 3.163 \cdot 6.45 \cdot 1.55 = 100.0$$

Continued..

- Is there a gain corresponding to a damping ratio of 0.707 or more for all system modes?

$\zeta = 0.707 = \cos(\theta)$

$\theta = 45^{\circ}$



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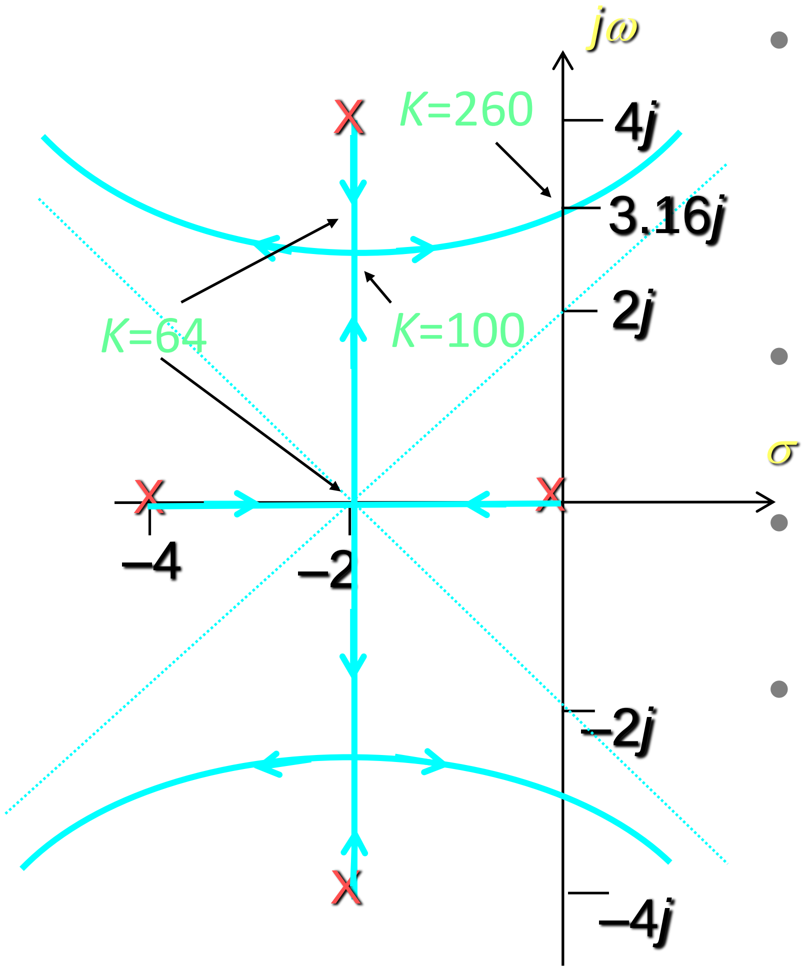
Time Responses



- Examine the responses for the various gains and relate them to the location of the closed-loop roots.
- K = 64, roots are –2, –2, –2±3.46j
- K = 100, roots are –2±2.45j, –2±2.45j
- K = 260, roots are ±3.16j, –4±3.16j